AN APPLICATION OF SPANNING TREES TO k-POINT SEPARATING FAMILIES OF FUNCTIONS

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Abstract

A family $\mathscr{F}$ of functions from $\mathbb{R}^n$ to $\mathbb{R}$ is k-point separating if, for every k-subset S of $\mathbb{R}^n$, there is a function $f \in \mathscr{F}$ such that f is one-to-one on S. The paper shows that, if the functions are required to be linear (or smooth), then a minimum k-point separating family $\mathscr{F}$ has cardinality $n(k-1)$. In the linear case, this result is extended to a larger class of fields including all infinite fields as well as some finite fields (depending on k and n). Also, some partial results are obtained for continuous functions on $\mathbb{R}^n$, including the case when k is infinite. The proof of the main result is based on graph theoretic results that have some interest in their own right. Say that a graph is an n-tree if it is a union of n edge-disjoint spanning trees. It is shown that every graph with $k \geqslant 2$ vertices and $n(k-1)$ edges has a non-trivial subgraph which is an n-tree. A determinantal criterion is also established for a graph with k vertices and $n(k-1)$ edges to be an n-tree.

1. *Introduction*

A family $\mathscr{F}$ of functions from $\mathbb{R}^n$ to $\mathbb{R}$ is *k-point interpolating* if, for every k-subset S of $\mathbb{R}^n$ and every function $f: S \to \mathbb{R}$, there is a function $f' \in \mathscr{F}$ such that $f'|_S = f$. An unsolved problem of approximation theory [**5**] is the determination of the smallest dimension $CI_k(\mathbb{R}^n, \mathbb{R})$ of a k-point interpolating subspace $\mathscr{F}$ of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$. This problem is unsolved even for $n = 2$. We investigate a related problem that was inspired by this one, namely, suppose that we have the less ambitious goal of obtaining a family $\mathscr{F}$ of functions such that, for each k-subset S of $\mathbb{R}^n$, there is a function $f \in \mathscr{F}$ such that f is injective on S. How small can $\mathscr{F}$ be?

We say that a function $f: X \to Y$ *separates* a subset S of X if f is injective on S. A family $\mathscr{F}$ of functions from X to Y is *k-point separating* if every k-subset S of X is separated by some $f \in \mathscr{F}$. Our principal result is the determination of the cardinality $LS_k(\mathbb{R}^n, \mathbb{R})$ (respectively, $DS_k(\mathbb{R}^n, \mathbb{R})$) of a smallest k-point separating family of linear (respectively, smooth) functions from $\mathbb{R}^n$ to $\mathbb{R}$. We prove that, if $n, k \geqslant 2$, then

$$LS_k(\mathbb{R}^n, \mathbb{R}) = DS_k(\mathbb{R}^n, \mathbb{R}) = n(k-1).$$

Our proof that $n(k-1)$ is an upper bound for $LS_k(\mathbb{R}^n, \mathbb{R})$ is based on graph theoretic results presented in Section 2 which have some interest in their own right. We say that a graph is an *n-tree* if it is a union of n edge-disjoint spanning trees. Such a graph with k vertices clearly has $n(k-1)$ edges. Using a result of Nash-Williams [**10**, **11**], we show that every graph with k vertices and $n(k-1)$ edges has a non-trivial subgraph which is an n-tree. We also establish a determinantal criterion for a graph with k vertices and $n(k-1)$ edges to be an n-tree.

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In Section 3, we prove that $LS_k(F^n, F) = n(k-1)$ holds for a large class of fields F, including all infinite fields and even some finite fields (depending on n and k). This result implies in particular that $DS_k(\mathbb{R}^n, \mathbb{R}) \leqslant n(k-1)$. Thus, to prove equality here, we only need to establish that $n(k-1) \leqslant DS_k(\mathbb{R}^n, \mathbb{R})$, which we do in Section 4. Finally, in Section 5, we prove some results for $CS_k(\mathbb{R}^n, \mathbb{R})$, the cardinality of a smallest k-point separating family of continuous functions. We find that $CS_k(\mathbb{R}^2, \mathbb{R}) \geqslant k$ for $k \geqslant 2$, and that $CS_k(\mathbb{R}^n, \mathbb{R}) > \lfloor k/2 \rfloor (n-1)$ for $n, k \geqslant 2$. We also consider the case in which k is infinite. We prove that, for any ordinal number α such that $\aleph_\alpha < |\mathbb{R}|$, if n is finite, then $CS_{\aleph_\alpha}(\mathbb{R}^n, \mathbb{R}) = LS_{\aleph_\alpha}(\mathbb{R}^n, \mathbb{R}) = \aleph_{\alpha+1}$.

2. *Graph theoretic results*

2.1. *n-trees*

In this section, $G = (V, E)$ denotes a graph with vertex set V and edge set E. We allow multiple edges but no loops in our graphs. Recall from graph theory that a *tree* is a connected acyclic graph. We say that G is an *n-tree* if there is a partition $\{E_1, \ldots, E_n\}$ of E such that, for every i, the graph (V, E_i) is a spanning tree of G, that is, G is a union of n edge-disjoint spanning trees. (The concept of an n-tree, although not the name, was introduced by Nash-Williams [**10**, **11**]. Our notion of an n-tree should not be confused with that of Harary and Palmer [**7**, 3.5, p. 73], which is an entirely different concept.)

We will need a theorem of Nash-Williams. Recall that a *forest* is an acyclic graph, namely a disjoint union of trees; a forest is connected if and only if it is a tree. For every subset $X \subseteq V$, let E_X denote the set of all edges of G both ends of which lie in the set X.

NASH-WILLIAMS THEOREM ([**11**]). *A graph $G = (V, E)$ is decomposable into n forests if and only if, for every non-empty subset X of V, we have*

$$|E_X| \leqslant n(|X|-1). \tag{1}$$

THEOREM 1. *If $|E| = n(|V|-1)$ and $|V| \geqslant 2$, then the graph $G = (V, E)$ contains a subgraph with at least two vertices which is an n-tree.*

Proof. Our theorem follows by induction on $|V|$. If $|V| = 2$, then E contains n edges. Each edge joins the two vertices in V, and therefore determines a spanning tree. Therefore, G is a union of n spanning trees and is therefore itself an n-tree.

Assume that $|V| \geqslant 3$, and that the theorem holds for graphs with fewer than $|V|$ vertices. Suppose that there is a proper subset X of V with $|X| \geqslant 2$ and $|E_X| \geqslant n(|X|-1)$. Then, by removing some edges from E_X if necessary, we obtain a subgraph (X, E') of (V, E) satisfying $|E'| = n(|X|-1)$ and $|X| < |V|$, and by induction we are done. Otherwise, for all proper subsets X of V with $|X| \geqslant 2$, we have $|E_X| < n(|X|-1)$. By hypothesis, (1) holds when $X = V$, and, since we have no loops, (1) holds trivially when $|X| = 1$. Hence (1) holds for all subsets X of V. We conclude from the Nash-Williams Theorem that G decomposes into n forests, say $F_i = (V_i, E_i)$, $i = 1, \ldots, n$. This means that $V = V_1 \cup \ldots \cup V_n$, $E = E_1 \cup \ldots \cup E_n$, and $E_i \cap E_j = \varnothing$ for $i \neq j$. A forest F_i with ω_i components and p_i vertices has $p_i - \omega_i$ edges. Since the edges of the forest partition the edges of G, we have

$$n(|V|-1) = (p_1 - \omega_1) + \ldots + (p_n - \omega_n).$$

Since $p_i \leqslant |V|$ and $\omega_i \geqslant 1$, it is clear that $p_i = |V|$ and $\omega_i = 1$ for all i. Hence each F_i is a spanning tree and consequently G is itself an n-tree. By induction, we are done, and the proof is complete.

2.2. *A determinantal criterion for n-trees*

We let $G = (V, E)$ be a graph with $p \geqslant 2$ vertices and q edges. We assume that $q = n(p-1)$, $n \geqslant 1$. This is obviously necessary for G to be an n-tree. We fix an ordering

$$v_1, v_2, \ldots, v_p$$

for the vertices of G, and an ordering

$$e_1, e_2, \ldots, e_q$$

for the edges of G. For each such graph, we define a $q \times q$ matrix $M(G)$ with entries which are polynomials in several variables over the rationals so that $\det M(G) \neq 0$ if and only if G is an n-tree. In the case in which G is an n-tree, the polynomial $\det M(G)$ is a kind of generating function for the ordered partitions of G into n spanning subtrees.

First we define the $q \times p$ matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & e_i \text{ has endpoints } v_j, v_k \text{ where } j < k \\ -1 & e_i \text{ has endpoints } v_j, v_k \text{ where } k < j \\ 0 & \text{otherwise.} \end{cases}$$

Fix $l \in [p] = \{1, \ldots, p\}$. Let K be the $q \times (p-1)$ matrix obtained by deleting the lth column of A. We see in Lemma 1 that the choice of l is not important for our purposes. Lemma 1 is well known (see, for example, [**1**, Propositions 5.3 and 5.4]), and so we omit its proof.

LEMMA 1. *Let T be any $(p-1)$-subset of $[q]$. Let K_T denote the $(p-1) \times (p-1)$ submatrix of K the row indices of which are in T. Let G_T be the spanning subgraph of G with edge set $\{e_i \mid i \in T\}$. Then* $\det K_T \in \{0, 1, -1\}$ *and* $\det K_T = \pm 1$ *if and only if G_T is a spanning tree in G. In particular, $|\det K_T|$ is independent of l.*

Let $\{m_j^i \mid i = 1, \ldots, n, j = 1, \ldots, q\}$ be a set of nq real numbers which are algebraically independent over the field $\mathbb{Q}$ of rational numbers. Set

$$X_i = \operatorname{diag}[m_1^i, m_2^i, \ldots, m_q^i] \tag{2}$$

for $i = 1, 2, \ldots, n$, and define $M = M(G)$ to be the matrix in block form:

$$M = [X_1 K \mid X_2 K \mid \ldots \mid X_n K]. \tag{3}$$

Here G is a graph with p vertices and $q = n(p-1)$ edges, and K is the $q \times (p-1)$ matrix defined above. Hence $X_i K$ is a $q \times (p-1)$ matrix, and M is a $q \times q$ matrix. Note that each row of A and therefore of K corresponds to an edge of G, and so, to simplify notation, we identify the edge e_i with row index i of K. We say that the $(p-1)$-set $T \subseteq [q] = E$ is a tree if the edge-induced subgraph is a tree.

Let $(T_1, T_2, \ldots, T_n)$ be an ordered partition of E such that each T_i is a spanning tree of G. Let

$$T_i = \{j_i^1, j_i^2, \ldots, j_i^{p-1}\}.$$

Form the monomial

$$m(T_1, T_2, \dots, T_n) = \prod_{i=1}^{n} m_{j_i^1}^{i} m_{j_i^2}^{i} \dots m_{j_i^{p-1}}^{i}.$$

For distinct partitions, we obtain distinct monomials; hence these monomials are linearly independent over $\mathbb{Q}$.

THEOREM 2. *If* G *is a graph with* p *vertices and* $q = n(p-1)$ *edges, then* $\det M(G) \neq 0$ *if and only if* G *is an* n*-tree. If* G *is an* n*-tree, then*

$$\det M(G) = \sum_{(T_1, T_2, \dots, T_n)} \varepsilon(T_1, T_2, \dots, T_n)\, m(T_1, T_2, \dots, T_n) \tag{4}$$

where $(T_1, T_2, \dots, T_n)$ *runs over all ordered partitions of* E *into disjoint spanning trees, and* $\varepsilon(T_1, T_2, \dots, T_n) \in \{1, -1\}$ *for all* $(T_1, T_2, \dots, T_n)$.

Proof. This theorem is a consequence of Lemma 1 and the following generalization of Laplace's expansion for determinants.

Let M be any $q \times q$ matrix and let $R, C \subseteq [q]$. Then $M_{R,C}$ denotes the submatrix of M with rows that have indices in R and columns that have indices in C. Laplace's expansion of $\det M$ can for our purposes be stated as follows (see, for example, [**9**, Theorem 12, p. 564]). For a fixed ordered partition (C_1, C_2) of $[q]$,

$$\det M = \sum_{(R_1, R_2)} \varepsilon(R_1, R_2, C_1, C_2) \det M_{R_1, C_1} \det M_{R_2, C_2} \tag{5}$$

where (R_1, R_2) runs over all ordered partitions of $[q]$ into two sets such that $|R_i| = |C_i|$, $i = 1, 2$, and $\varepsilon(R_1, R_2, C_1, C_2) \in \{1, -1\}$. This generalizes by induction to the following. If $(C_1, \dots, C_n)$ is any fixed partition of $[q]$, then

$$\det M = \sum_{(R_1, \dots, R_n)} \varepsilon(R_1, \dots, R_n, C_1, \dots, C_n) \det M_{R_1, C_1} \dots \det M_{R_n, C_n} \tag{6}$$

where $(R_1, R_2, \dots, R_n)$ runs over all ordered partitions of $[q]$ into n sets such that $|R_i| = |C_i|$ for all i and $\varepsilon(R_1, \dots, R_n, C_1, \dots, C_n) \in \{1, -1\}$.

To apply this to the $q \times q$ matrix

$$M = [X_1 K \,|\, X_2 K \,|\, \dots \,|\, X_n K],$$

we take

$$\begin{aligned} C_1 &= \{1, 2, \dots, p-1\} \\ C_2 &= \{p, p+1, \dots, 2(p-1)\} \\ &\dots \\ C_n &= \{(n-1)(p-1)+1, \dots, n(p-1)\}. \end{aligned}$$

Note that C_i has $p-1$ elements corresponding to the column indices of $X_i K$. Now, if $R_i \subseteq [q] = E$ is a $(p-1)$-set with $R_i = \{j_1, \dots, j_{p-1}\}$, we obtain

$$\det M_{R_i, C_i} = \det(\mathrm{diag}[m_{j_1}^i, \dots, m_{j_{p-1}}^i]\, K_{R_i}).$$

Hence, by Lemma 1,

$$\det M_{R_i, C_i} = \begin{cases} \pm m_{j_1}^i \dots m_{j_{p-1}}^i & R_i \text{ is a spanning tree} \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

From (6) and (7), we immediately obtain (4). Then, since the monomials $m(T_1, \ldots, T_n)$ are linearly independent, $\det M(G) \neq 0$ if and only if G is an n-tree.

In this paper, we are primarily interested in families of functions from $\mathbb{R}^n$ to $\mathbb{R}$. However, some of our results hold when $\mathbb{R}$ is replaced by other fields. For background material on field theory, see, for example, [**8**].

We now introduce a definition that allows us to extend our results to a wider class of fields. The elements m_j^i can be replaced by field elements which are not algebraically independent over the ground field.

DEFINITION 1. Let X be a finite subset of a field F. For each subset S of X, define

$$\pi(S) = \begin{cases} \prod_{s \in S} s & S \neq \varnothing \\ 1 & S = \varnothing. \end{cases}$$

Let

$$\binom{X}{k}$$

denote the set of all k-subsets of X. We say that X is *k-Boolean independent* if, for all functions

$$\alpha : \binom{X}{k} \longrightarrow \{0, 1, -1\},$$

$$\sum_{S \in \binom{X}{k}} \alpha(S)\,\pi(S) = 0 \quad \Rightarrow \quad \alpha(S) = 0 \quad \text{for all } S \in \binom{X}{k}.$$

Note that, if X is k-Boolean independent and $k' \leqslant k$, then X may not be k'-Boolean independent.

EXAMPLE 1. Let $F = K(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are algebraically independent over K. In this case, $X = \{x_1, \ldots, x_n\}$ is k-Boolean independent for $1 \leqslant k \leqslant n$.

EXAMPLE 2. Let F be a field and let

$$F_0 \leqslant F_1 \leqslant F_2 \leqslant \ldots \leqslant F_t = F$$

be a chain of subfields, where $F_i = F_{i-1}(\theta_i)$ for $i = 1, \ldots, t$ and $[F_i : F_{i-1}] = n_i \geqslant 2$. Let $X = \{\theta_1, \ldots, \theta_t\}$. Since, for each i, $\{1, \theta_i, \theta_i^2, \ldots, \theta_i^{n_i - 1}\}$ is a basis for F_i over F_{i-1}, it follows that $\{\pi(S) \mid S \subseteq X\}$ is linearly independent over F_0. In particular, X is k-Boolean independent for all $1 \leqslant k \leqslant t$. The following are examples:

(1) Let $F = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n})$ where $p_1, p_2, \ldots, p_n$ are distinct primes. The set $X = \{\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n}\}$ is k-Boolean independent for $1 \leqslant k \leqslant n$ since, for each i, $\sqrt{p_i} \notin \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_{i-1}})$ (see [**12**]).

(2) Let $F = GF(p^n)$ where p is any prime and $n = n_1 n_2 \ldots n_t$, $n_i \geqslant 2$, for all i. Then there is a tower of subfields

$$F_0 \leqslant F_1 \leqslant F_2 \leqslant \ldots \leqslant F_t = F$$

where $F_0 = GF(p)$, $F_1 = GF(p^{n_1})$, $F_2 = GF(p^{n_1 n_2}), \ldots, F_t = GF(p^n)$. Then, for $i = 1, \ldots, t$, there exists a $\theta_i \in F$ such that $F_i = F_{i-1}(\theta_i)$. Hence $X = \{\theta_1, \ldots, \theta_t\}$ is k-Boolean independent for $1 \leqslant k \leqslant t$.

EXAMPLE 3. For integers $n \geqslant k \geqslant 1$, let $X = \{m_1, \dots, m_n\}$ be a set of positive integers chosen so that

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} m_i^k < m_{i+1}$$

for $i = 1, 2, \dots, n-1$. It is easy to prove by induction on t that X is a t-Boolean independent subset of $\mathbb{Q}$ whenever $1 \leqslant t \leqslant k$.

EXAMPLE 4. Let F contain an element x which is transcendental over the prime field of F. Let $m_1, m_2, \dots, m_n$ be a sequence of positive integers satisfying

$$m_1 + m_2 + \dots + m_i < m_{i+1} \quad \text{for } 2 \leqslant i \leqslant n-1.$$

It is easy to see that any integer $m = m_{i_1} + \dots + m_{i_k}$, where $i_1 < \dots < i_k$, uniquely determines the summands $m_{i_1}, \dots, m_{i_k}$, and, since the powers of x are linearly independent over the prime field, it follows that the set $X = \{x^{m_1}, x^{m_2}, \dots, x^{m_n}\}$ is k-Boolean independent for $1 \leqslant k \leqslant n$.

PROPOSITION 1. *If F is an infinite field and n is a positive integer, then F contains an n-subset which is k-Boolean independent for $1 \leqslant k \leqslant n$.*

Proof. If F has characteristic 0, then $\mathbb{Q}$ is a subfield of F, and the result follows from Example 3. Suppose that F has characteristic p. If F is algebraic over $\mathbb{Z}_p$, then, for any finite subfield K of F, take an element $\alpha \in F - K$. Now α is algebraic over K since $\mathbb{Z}_p \subseteq K$. Hence $K(\alpha)$ is a finite subfield of F that contains K properly. Thus we can produce an arbitrarily long chain of finite subfields of F, and the proposition follows from Example 2. If F is not algebraic over $\mathbb{Z}_p$, then F contains an element which is transcendental over $\mathbb{Z}_p$, and the result follows from Example 4.

COROLLARY 1. *Theorem 2 holds if in the definition of $M(G)$ we replace the set of real numbers m_j^i by any set of $n(p-1)$-Boolean independent elements of cardinality $n^2(p-1)$ in any field.*

3. *k-point separation with linear functionals*

Let U be a vector space over a field F, and let U^* be the dual space of U. We define

$$LS_k(U, F) = \min\{|\mathscr{F}| : \mathscr{F} \subseteq U^* \text{ and } \mathscr{F} \text{ is } k\text{-point separating}\}. \tag{8}$$

We say that $LS_k(U, F)$ is undefined if there is no k-point separating family of linear functionals in the dual space of U. For any field F, n-dimensional vector space U over F and positive integer k, it is shown in [**2**] that, if

$$(n-1)\binom{k}{2} \leqslant |F|,$$

then $LS_k(U, F)$ is well defined and

$$LS_k(U, F) \leqslant (n-1)\binom{k}{2} + 1.$$

From this we see that $LS_k(U, F)$ can be undefined only when F is finite. This is the case, for example, when $k > |F|$. For more details on the case when F is finite, see [**2**].

THEOREM 3. *Let F be a field which contains a set of cardinality $n^2(k-1)$ which is t-Boolean independent for $2 \leqslant t \leqslant n(k-1)$, and let U be an n-dimensional vector space over F. Then, for $n, k \geqslant 2$, we have $LS_k(U, F) = n(k-1)$.*

We divide the proof of Theorem 3 into two lemmas.

LEMMA 2. *If F and U are as in Theorem 3, then, for $n, k \geqslant 2$, $LS_k(U, F)$ is defined and $LS_k(U, F) \leqslant n(k-1)$.*

Proof. Let $\{m_j^i \mid j = 1, \ldots, n(k-1),\ i = 1, \ldots, n\}$ be a set of cardinality $n^2(k-1)$ in F which is t-Boolean independent for $2 \leqslant t \leqslant n(k-1)$. Take $U = F^n$, and, for each $j = 1, \ldots, n(k-1)$, define $f_j \in U^*$ by

$$f_j(x_1, x_2, \ldots, x_n) = m_j^1 x_1 + m_j^2 x_2 + \ldots + m_j^n x_n.$$

We claim that these $n(k-1)$ linear functionals are k-point separating on U. Suppose that this is not true. Then there is a set $V = \{p_1, \ldots, p_k\}$ of k distinct points in U such that, for each j, f_j restricted to V is not injective. We construct a graph G with vertex set V and edge set $E = \{f_1, \ldots, f_{n(k-1)}\}$. The incidence relation is defined as follows. For each $j \in \{1, 2, \ldots, n(k-1)\}$, since f_j is not injective on V, we can select integers a_j, b_j such that $1 \leqslant a_j < b_j \leqslant k$ and

$$f_j(p_{a_j}) = f_j(p_{b_j}). \tag{9}$$

We say that p_{a_j} and p_{b_j} are endpoints of the edge f_j. Note that different edges may join the same vertices, but there are no loops.

This gives us a graph G with k vertices and $n(k-1)$ edges. By Theorem 1, G contains a subgraph $H = (V_H, E_H)$ with at least two vertices which is an n-tree. By renumbering, we can take

$$V_H = \{p_1, \ldots, p_t\} \quad 2 \leqslant t \leqslant k$$

and

$$E_H = \{f_1, \ldots, f_q\} \quad q = n(t-1).$$

Write

$$p_a = (x_1^a, x_2^a, \ldots, x_n^a) \quad a = 1, 2, \ldots, t.$$

Now, in coordinates, (9) becomes, for $j \in [q]$,

$$m_j^1 x_1^{a_j} + m_j^2 x_2^{a_j} + \ldots + m_j^n x_n^{a_j} - m_j^1 x_1^{b_j} - m_j^2 x_2^{b_j} - \ldots - m_j^n x_n^{b_j} = 0. \tag{10}$$

We think of (10) as a homogeneous system of $n(t-1)$ linear equations in nt variables x_j^i. Since the p_i are distinct and the f_j are linear, we can replace p_i by $p_i' = p_i - p_t$, for all i. Then $p_t' = 0$, and so we have reduced the system to a system of $n(t-1)$ equations in $n(t-1)$ variables $z_j^i = x_j^i - x_j^t$, $i = 1, \ldots, t-1$, $j = 1, \ldots, n$.

Now, if we order the variables

$$z_1^1, z_1^2, \ldots, z_1^{t-1}, z_2^1, z_2^2, \ldots, z_2^{t-1}, \ldots, z_n^1, z_n^2, \ldots, z_n^{t-1},$$

then the matrix of coefficients of the linear system is equal to the matrix $M(H)$ of Theorem 2. By Corollary 1, since H is an n-tree, we have $\det M(H) \neq 0$. This implies that the homogeneous system has only the trivial solution. It follows that $p_i' = 0$ for all i. Hence $p_i = p_t$ for all i. This contradicts the fact that the p_i are distinct. The proof is complete.

We recall some definitions and results from linear algebra. Let U be an n-dimensional vector space over a field F. An *affine subspace X of dimension w* in U is a coset $X = x+W$ where W is a (linear) subspace of U of dimension w. X is a *hyperplane* (respectively, *line*) if X is of dimension $n-1$ (respectively, 1). The following are well known:

(A) If L is a line and H is a hyperplane in U, then either (i) L is contained in some (unique) hyperplane parallel to H, or (ii) L intersects each hyperplane parallel to H in precisely one point.

(B) If $H_1, \dots, H_t$ are hyperplanes in U, then the intersection $H_1 \cap \dots \cap H_t$ is either empty or an affine subspace of dimension at least $n-t$.

Let $\mathscr{F}$ be a family of functions from X to Y, and let $S \subseteq X$. We say that S *spoils* $\mathscr{F}$ if, for each $f \in \mathscr{F}$, there exist $x, y \in S$, $x \neq y$, such that $f(x) = f(y)$. Clearly $\mathscr{F}$ is k-point separating if and only if $\mathscr{F}$ is not spoiled by any subset S of X with $|S| \leqslant k$.

LEMMA 3. *If U is an n-dimensional vector space over a field F, then $LS_k(U,F) \geqslant n(k-1)$ provided that $LS_k(U,F)$ is defined. In particular, this is the case when F is infinite.*

Proof. Let $\mathscr{F}$ be any family of linear functionals on U of cardinality $n(k-1)-1$. It suffices to show that there is some subset S of U of cardinality less than or equal to k which spoils $\mathscr{F}$. The idea of our proof is first to find two points p, q which spoil at least $n-1$ functionals in $\mathscr{F}$. Then we generate successively additional points p_1, $p_2, \dots, p_l, \dots, p_{k-2}$ so that, as each new point p_l is generated, we make sure that n additional functionals in $\mathscr{F}$ are spoiled by the three points p, q, p_l. Then, since $(n-1)+(k-2)n = n(k-1)-1$, all of $\mathscr{F}$ is spoiled by $\{p, q, p_1, p_2, \dots, p_{k-2}\}$.

Clearly we can assume that each functional $f \in \mathscr{F}$ is not zero, and so $f^{-1}(c)$ is a hyperplane for $c \in F$. We begin by selecting any $(n-1)$-subset $\mathscr{F}_0'$ of $\mathscr{F}$. Then, since $0 \in f^{-1}(0)$ for all $f \in \mathscr{F}_0'$, we have by result (B) above that $I = \bigcap_{f \in \mathscr{F}_0'} f^{-1}(0)$ has dimension at least 1. Let $p, q \in I$, $p \neq q$, and let L_0 be the line containing p and q. Let

$$\mathscr{F}_0 = \{f \in \mathscr{F} \mid L_0 \subseteq f^{-1}(c) \text{ for some } c \in F\}.$$

Note that $\mathscr{F}_0$ contains at least $n-1$ functionals in $\mathscr{F}$ and is spoiled by $\{p, q\}$. Now assume that we have found, for each $l < k-2$, the following:

(1) a subset $\mathscr{F}_l$ of $\mathscr{F}$ such that (i) $\mathscr{F}_0 \subseteq \mathscr{F}_l$ and (ii) $|\mathscr{F}_l| \geqslant (n-1)+ln$;

(2) a subset $\{p, q, p_1, \dots, p_l\}$ of U which spoils $\mathscr{F}_l$.

If $|\mathscr{F}_l| \geqslant n-1+(l+1)n$, we take $\mathscr{F}_{l+1} = \mathscr{F}_l$ and $p_{l+1} = p_l$. Thus we can assume that

$$|\mathscr{F}_l| < (n-1)+(l+1)n \leqslant n(k-1)-1.$$

We select $\mathscr{D}' \subseteq \mathscr{F} \setminus \mathscr{F}_l$ such that $|\mathscr{D}'| = n-1$ if $|\mathscr{F} \setminus \mathscr{F}_l| \geqslant n-1$; otherwise, take $\mathscr{D}' = \mathscr{F} \setminus \mathscr{F}_l$. Then the intersection $J = \bigcap_{f \in \mathscr{D}'} f^{-1}(f(p))$ contains p, and, by result (B) above, it has dimension at least 1. Therefore, there is a line $L_1 \subseteq J$ with $p \in L_1$. Let

$$\mathscr{D} = \{f \in \mathscr{F} \mid L_1 \subseteq f^{-1}(c) \text{ for some } c \in F\}.$$

Now there are two cases: (I) $\mathscr{F}_l \cup \mathscr{D} = \mathscr{F}$, and (II) $\mathscr{F}_l \cup \mathscr{D} \neq \mathscr{F}$. In case I, let p_{l+1} be any point on line L_1 that is different from p. Then $\mathscr{D}$ is spoiled by $\{p, p_{l+1}\}$, and hence all of $\mathscr{F}$ is spoiled by $\{p, q, p_1, \dots, p_{l+1}\}$. In case II, take $g \in \mathscr{F} \setminus (\mathscr{F}_l \cup \mathscr{D})$, and let $d = g(q)$. Now, by the definition of $\mathscr{D}$, L_1 is not contained in any hyperplane $g^{-1}(c)$ parallel to $g^{-1}(d)$, and so, by result (A) above, $g^{-1}(d)$ intersects L_1 in exactly one point; call this p_{l+1}. We claim that $p_{l+1} \neq p$ and $p_{l+1} \neq q$. If $p_{l+1} = p$, then the line $L_0 = \overline{pq}$ lies in

$g^{-1}(d)$; this implies that $g \in \mathscr{F}_0$, but $\mathscr{F}_0 \subseteq \mathscr{F}_l$ and $g \notin \mathscr{F}_l$. If $p_{l+1} = q$, the lines $L_0 = \overline{pq}$ and $L_1 = \overline{pp_{l+1}}$ are equal. This implies that $\mathscr{D} = \mathscr{F}_0$. Then $\mathscr{D}' \subseteq \mathscr{D} \subseteq \mathscr{F}_0 \subseteq \mathscr{F}_l$, which is contrary to the definition of $\mathscr{D}'$. It follows that $\mathscr{F}_{l+1} = \mathscr{F}_l \cup \mathscr{D} \cup \{g\}$ is spoiled by $\{p, q, p_1, \ldots, p_{l+1}\}$ and $|\mathscr{F}_{l+1}| \geqslant (n-1)+ln+(n-1)+1 = n-1+(l+1)n$, as desired.

4. *k-point separation with smooth functions*

In this section, we determine the cardinality $DS_k(\mathbb{R}^n, \mathbb{R})$ of a smallest family of smooth ($\mathscr{C}^\infty$) functions from $\mathbb{R}^n$ to $\mathbb{R}$ that separates every set of k distinct points in $\mathbb{R}^n$. Our references for this section are [**6**, **14**, **15**]. Given

$$f_1, \ldots, f_m : \mathbb{R}^n \longrightarrow \mathbb{R},$$

we write $F = (f_1, \ldots, f_m)$ for the mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$F(x) = (f_1(x), \ldots, f_m(x))$$

for $x \in \mathbb{R}^n$. We identify the derivative $DF(x) : \mathbb{R}^n \to \mathbb{R}^m$ with the Jacobian matrix $[\partial f_i(x)/\partial x_j]$. The rank of F at x is the rank of $DF(x)$. We require three lemmas.

LEMMA 4. *If $F : \mathbb{R}^n \to \mathbb{R}^m$ is smooth on an open set U, then there is a non-empty open set $V \subseteq U$ on which F has constant rank.*

Proof. Let x be a point of U at which the rank of $DF(x)$ attains its maximum, say k. This implies that $DF(x)$ has a $k \times k$ non-singular submatrix. Since the determinant is a continuous function, it follows that the same submatrix is non-singular in a neighborhood V of x, and hence $DF(X)$ has rank k in V.

LEMMA 5 ([**15**, Proposition 12, p. 65]). *If $F : \mathbb{R}^n \to \mathbb{R}^m$ is smooth and of constant rank k on an open set U, then, for each $x \in U$, $M = U \cap F^{-1}(F(x))$ is a manifold of dimension $n-k$.*

In Lemma 5, the tangent space to M at x is the kernel of $DF(x)$.

LEMMA 6 ([**14**, Theorem 5-2, p. 111]). *A subset M of $\mathbb{R}^n$ is a k-dimensional manifold if and only if for each $x \in M$ there is an open set U of $\mathbb{R}^n$ containing x, an open set W in $\mathbb{R}^k$, and a* 1-1 *smooth function $f : W \to \mathbb{R}^n$ such that $f(W) = M \cap U$, and $Df(y)$ has rank k for each $y \in W$.*

The function f in Lemma 6 is called a coordinate system around x. In this case, the tangent space to M at x is the image of $Df(y)$, where $f(y) = x$.

THEOREM 4. *For all $n \geqslant 1$ and $k \geqslant 2$, $DS_k(\mathbb{R}^n, \mathbb{R}) = n(k-1)$.*

Proof. Since linear functionals on $\mathbb{R}^n$ are smooth, by Proposition 1 and Lemma 2,

$$DS_k(\mathbb{R}^n, \mathbb{R}) \leqslant n(k-1).$$

It therefore suffices to prove that any family $\mathscr{F}$ of smooth functions on $\mathbb{R}^n$ of cardinality $nk-n-1$ is spoiled by some set of at most k points in $\mathbb{R}^n$.

For $\mathscr{G} \subseteq \mathscr{F}$, let $g_1, \ldots, g_s$ be the distinct elements of $\mathscr{G}$ listed in any fixed order. Define

$$F_{\mathscr{G}} : \mathbb{R}^n \longrightarrow \mathbb{R}^s \quad \text{by} \quad F_{\mathscr{G}} = (g_1, \ldots, g_s).$$

Note that

$$y \in F_{\mathscr{G}}^{-1}(F_{\mathscr{G}}(x)) - \{x\} \quad \Leftrightarrow \quad \{x, y\} \text{ spoils } \mathscr{G}.$$

By Lemma 4, since $\mathscr{F}$ is finite, we can find a non-empty open set U of $\mathbb{R}^n$ such that, for all $\mathscr{G} \subseteq \mathscr{F}$, $F_{\mathscr{G}}$ has constant rank throughout U. Without loss of generality, we take $U = \mathbb{R}^n$.

Our plan is to show that there exists a partition of $\mathscr{F}$

$$\mathscr{F} = \mathscr{F}_0 \cup \mathscr{F}_1 \cup \ldots \cup \mathscr{F}_{k-2} \tag{11}$$

such that the following are true:

(1) $|\mathscr{F}_0| = n-1$ and $|\mathscr{F}_j| = n$ for $1 \leqslant j \leqslant k-2$;

there exists a point $a \in \mathbb{R}^n$, and there exists a nested sequence L_j, $0 \leqslant j \leqslant k-2$, of submanifolds of $\mathbb{R}^n$ of positive dimension such that, for $1 \leqslant j \leqslant k-2$,

(2) $a \in L_j \subseteq L_{j-1}$, and, for each $b \in L_j - \{a\}, \{a, b\}$ spoils $\mathscr{F}_0$, and, if $j \geqslant 1$, there is a point $c_j = c_j(b) \in \mathbb{R}^n$ such that $\{a, b, c_j\}$ spoils $\mathscr{F}_j$.

Once we have established the existence of such a partition, we can complete the proof by taking $b \in L_{k-2} - \{a\}$. Then $b \in L_i - \{a\}$ for all i, and so, by (2) above, there exists for each $i \geqslant 1$ a point c_i such that $\{a, b, c_i\}$ spoils $\mathscr{F}_i \cup \mathscr{F}_0$. Hence $\{a, b, c_1, c_2, \ldots, c_{k-2}\}$ spoils $\mathscr{F}$, as desired.

To prove the existence of such a partition, we consider first the easy case in which rank $(F_{\mathscr{G}}) < n-1$ for every $(n-1)$-subset $\mathscr{G}$ of $\mathscr{F}$. In this case, we can take (11) to be any partition satisfying (1) above, and let a be any point of $\mathbb{R}^n$. Write $F_i = F_{\mathscr{F}_i}$ for all i, and set

$$L_i = F_0^{-1} F_0(a) \quad 0 \leqslant i \leqslant k-2.$$

Since rank $(F_0) < n$, by Lemma 5, L_i is a manifold of positive dimension. Hence, if $b \in L_i - \{a\}$, then $\{a, b\}$ spoils $\mathscr{F}_0$. Under our current assumptions for $i \geqslant 1$, rank$(F_i) < n$, and so $M_i = F_i^{-1} F_i(a)$ is also a manifold of positive dimension; if $c_i \in M_i - \{a\}$, then $\{a, c_i\}$ spoils $\mathscr{F}_i$, and so $\{a, b, c_i\}$ also spoils $\mathscr{F}_i$.

It remains to consider the case in which there is an $(n-1)$-subset $\mathscr{G}_0$ of $\mathscr{F}$ with rank$(F_{\mathscr{G}_0}) = n-1$. In this case, in (11), take $\mathscr{F}_0 = \mathscr{G}_0$, and let the rest of the partition be chosen arbitrarily subject to (1) above. Again write $F_i = F_{\mathscr{F}_i}$, and let a be any point of $\mathbb{R}^n$. We define the L_i recursively as follows. First let L_0 be the 1-dimensional manifold $F_0^{-1} F_0(a)$. Clearly, if $b \in L_0 - \{a\}$, then $\{a, b\}$ spoils $\mathscr{F}_0$. Assume that we have already selected $L_j, 0 \leqslant j < i \leqslant k-2$, satisfying the conditions in (2) above. Notice that each L_j is 1-dimensional. To define L_i, we consider three cases:

Case 1: rank$(F_i) < n$.

In this case, we set $L_i = L_{i-1}$. Then $M = F_i^{-1} F_i(a)$ is a manifold of positive dimension, and so there is an element $c_i \in M - \{a\}$, and hence $\{a, c_i\}$ spoils $\mathscr{F}_i$.

Case 2: There exists an $f \in \mathscr{F}_i$ such that the mapping $F = F_{\mathscr{F}_0 \cup \{f\}}$ has rank $n-1$.

In this case, $M = F^{-1}F(a)$ is a manifold of dimension 1 containing a and contained in L_0. By assumption, $a \in L_{i-1} \subseteq L_0$. Since M and L_{i-1} are both 1-dimensional submanifolds contained in the 1-dimensional manifold L_0 and $a \in M \cap L_{i-1}$ using Lemma 6, one can show that $L_i := M \cap L_{i-1}$ is a 1-dimensional manifold. Note that L_i contains a and is contained in L_{i-1}. If $b \in L_i - \{a\}$, then $\{a, b\}$ spoils $\mathscr{F}_0$. Since $b \in M - \{a\}, \{a, b\}$ spoils $\mathscr{F}_0 \cup \{f\}$. On the other hand,

$$M' = F_{\mathscr{F}_i - \{f\}}^{-1}(F_{\mathscr{F}_i - \{f\}}(a))$$

is a manifold of positive dimension, and so any point $c_i \in M' - \{a\}$ makes $\{c_i, a\}$ spoil $\mathscr{F}_i - \{f\}$. Then $\{a, b, c_i\}$ spoils $\mathscr{F}_i$.

Case 3: $\operatorname{rank}(F_i) = n$ and $\operatorname{rank} F_{\mathscr{F}_0 \cup \{f\}} = n$ for all $f \in \mathscr{F}_i$.

Let

$$\mathscr{F}_0 = \{f_1, \ldots, f_{n-1}\} \quad \text{and} \quad \mathscr{F}_i = \{g_1, \ldots, g_{n-1}, g_n\}.$$

Then

$$F_0 = F_{\mathscr{F}_0} = (f_1, \ldots, f_{n-1}) \quad \text{and} \quad F_i = F_{\mathscr{F}_i} = (g_1, \ldots, g_{n-1}, g_n).$$

In addition, we define

$$F = F_{\mathscr{F}_0 \cup \{g_n\}} = (f_1, \ldots, f_{n-1}, g_n) \quad \text{and} \quad G = F_{\mathscr{F}_i - \{g_n\}} = (g_1, \ldots, g_{n-1}).$$

In this case, F_i and F have rank n, and F_0 and G have rank $n-1$. Thus the manifold $K = G^{-1}G(a)$ is 1-dimensional. Now it suffices to establish Assertion 1.

ASSERTION 1. There is a 1-dimensional manifold L_i with $a \in L_i \subseteq L_{i-1}$ and a 1-dimensional manifold $K_0 \subseteq K$ such that, for any $x \in L_i$, the level set $g_n^{-1} g_n(x)$ contains exactly one point from each of the manifolds L_i and K_0.

Once Assertion 1 has been established, we can complete the proof as follows. Let $b \in L_i - \{a\}$. Then $b \in L_0 - \{a\}$, and so $\{a, b\}$ spoils $\mathscr{F}_0$. By Assertion 1, $L_i \cap g_n^{-1} g_n(b) = \{b\}$ and $K_0 \cap g_n^{-1} g_n(b) = \{c\}$ for some point c. Now there are two possibilities: $b = c$, or $b \neq c$. Suppose that $b = c$. Then $b \in K_0 - \{a\}$, and so $\{a, b\}$ spoils $g_1, \ldots, g_{n-1}$. Then we take c_i to be any element of $g_n^{-1} g_n(b) - \{b\}$, and $\{b, c_i\}$ spoils g_n. Hence $\{a, b, c_i\}$ spoils $\mathscr{F}_i$. On the other hand, if $b \neq c$, then $\{b, c\}$ spoils g_n. Also, $c \in K_0 \subseteq G^{-1}G(a)$, and so $\{a, c\}$ spoils $g_1, \ldots, g_{n-1}$ unless $c = a$. However, if $c = a$, then a and b are distinct elements of $L_i \cap g_n^{-1} g_n(b)$, which contradicts Assertion 1. Therefore, if we set $c_i = c$, then $\{a, b, c_i\}$ spoils $\mathscr{F}_i$, and the proof is completed.

It remains to prove Assertion 1. Since F_i has rank n throughout $\mathbb{R}^n$, by the Inverse Function Theorem, there are open sets U and V with $a \in U$ such that F_i maps U diffeomorphically onto V. To simplify the proof, we identify U with V via F_i. This allows us to assume that the functions g_j are simply the coordinate projections $g_j(x_1, \ldots, x_n) = x_j$ for $(x_1, \ldots, x_n) \in U$.

From above, both K and L_0 are 1-dimensional manifolds containing a. Since L_{i-1} is a manifold of positive dimension contained in L_0, it also has dimension 1, and, by assumption, it contains a. It follows from Lemma 6 that there are coordinate systems

$$: I_1 \longrightarrow K \quad \text{and} \quad \psi : I_2 \longrightarrow L_{i-1}$$

where I_1 and I_2 are open intervals in $\mathbb{R}$ chosen so that $0 \in I_1 \cap I_2$,

$$(0) = \psi(0) = a = (a_1, \ldots, a_n)$$

and each of $D \ (t)$, $t \in I_1$, and $D\psi(t)$, $t \in I_2$, have rank 1, and in particular are non-zero. We write

$$(t) = (\ _1(t), \ldots, \ _n(t)) \quad \text{for } i \in I_1$$
$$\psi(t) = (\psi_1(t), \ldots, \psi_n(t)) \quad \text{for } i \in I_2.$$

Then

$$D \ (t) = (\ '_1(t), \ldots, \ '_n(t)) \quad \text{for } i \in I_1$$
$$D\psi(t) = (\psi'_1(t), \ldots, \psi'_n(t)) \quad \text{for } i \in I_2.$$

We wish to establish that $\ '_n(t) \neq 0$ for $t \in I_1$ and $\psi'_n(t) \neq 0$ for $t \in I_2$.

Note that, since g_n is the projection on the nth component, for all $y \in U$, the tangent space to $g_n^{-1} g_n(y)$ is the hyperplane of all points $(x_1, \dots, x_n) \in \mathbb{R}^n$ such that $x_n = 0$. Thus, to show that $\psi'_n(t) \neq 0$, it suffices to show that $D\psi(t)$ is not in the tangent space to $g_n^{-1} g_n(y)$ when $y = \psi(t)$.

Suppose that $v := D\psi(t)$ is in the tangent space to $g_n^{-1} g_n(y)$ and $y = \psi(t)$. By the comment following Lemma 5, we have $Dg_n(y)v = 0$. By the same comment, since v is in the tangent space to L_0, we have $DF_0(y)v = 0$. It follows that $DF(y)v = 0$. However, $DF(y)$ is non-singular since F has rank n and hence $v = 0$. This contradicts the fact that $D\psi(t)$ is non-zero for $t \in I_2$, and it thus proves that $\psi'_n(t) \neq 0$ for $t \in I_2$.

By the same reasoning, since rank $F_i = n$, we can prove that $\;'_n(t) \neq 0$ for all $t \in I_1$.

This shows that the mappings $\;_n : I_1 \to \mathbb{R}$ and $\psi_n : I_2 \to \mathbb{R}$ are both injections. Since $\;_n(0) = \psi_n(0) = a_n$, the nth component of a, there is an open interval I in $\mathbb{R}$ such that $a_n \in I \subseteq \;_n(I_1) \cap \psi_n(I_2)$. Let $J_1 = \;_n^{-1}(I)$ and $J_2 = \psi_n^{-1}(I)$. Finally, if we set $K_0 = \;(J_1)$ and $L_i = \psi(J_2)$, then Assertion 1 holds. This completes the proof.

5. *k-point separation with continuous functions*

The motivation for this paper was originally the following question. Suppose that $\mathscr{F}$ is a family of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ that separates every k-element set of points in $\mathbb{R}^n$. How small can $\mathscr{F}$ be? Unfortunately, we were unable to determine this number, $CS_k(\mathbb{R}^n, \mathbb{R})$, for all k, n. Nevertheless, we know from the linear case in Section 3 that, if $n, k > 1$, then $CS_k(\mathbb{R}^n, \mathbb{R}) \leqslant n(k-1)$, and we suspect that equality holds. All we can say at present is that the lower bound on $CS_k(\mathbb{R}^n, \mathbb{R})$ increases at least linearly with k and n.

PROPOSITION 2. *If $n > 1$, then $CS_k(\mathbb{R}^n, \mathbb{R}) \geqslant k$ for all $k > 0$.*

Proof. We prove the proposition by induction on k. As there is no continuous injection from $\mathbb{R}^n$ to $\mathbb{R}$, the proposition holds for $k = 2$. We claim that, if the proposition holds for k, then it holds for $k+1$. Since $CS_k(\mathbb{R}^n, \mathbb{R}) \geqslant CS_k(\mathbb{R}^2, \mathbb{R})$ for $n \geqslant 2$, we can assume that $n = 2$.

Suppose that $CS_k(\mathbb{R}^n, \mathbb{R}) \geqslant k$, and that $f_1, \dots, f_k : \mathbb{R}^n \to \mathbb{R}$ are continuous. We will find a $(k+1)$-set that $f_1, \dots, f_k$ fail to separate. By induction, there exists a k-set $\{p_1, \dots, p_k\} \subseteq \mathbb{R}^n$ that $f_1, \dots, f_{k-1}$ fails to separate. If this set is not separated by f_k either, we are done; otherwise, without loss of generality, we can suppose that $f_k(p_1) < f_k(p_2) < \dots < f_k(p_k)$. There are two cases.

Case 1: If $k = 2$, draw three disjoint arcs, and call them A_1, A_2, A_3, from p_1 to p_2. By the Intermediate Value Theorem, there exist $q_1 \in A_1$, $q_2 \in A_2$ and $q_3 \in A_3$ such that $f_2(q_1) = f_2(q_2) = f_2(q_3)$. If any two of $f_1(q_1), f_1(q_2), f_1(q_3)$ are equal, say $f_1(q_1) = f_1(q_2)$, then $\{q_1, q_2\}$ spoil f_1 and f_2. Thus, without loss of generality, suppose that $f_1(q_1) < f_1(q_2) < f_1(q_3)$. We have two subcases. If $f_1(q_2) = f_1(p_1)$, then $\{p_1, q_2, q_3\}$ spoil f_1 and f_2. On the other hand, if $f_1(q_2) \neq f_1(p_1)$, say $f_1(q_2) > f_1(p_1)$, then, by the Intermediate Value Theorem, there exists a $q'_3 \in A_3$ between p_1 and q_3 such that $f_1(q'_3) = f_1(q_2)$, and $\{q_2, q_3, q'_3\}$ spoils f_1, f_2; similarly, if $f_1(q_2) < f_1(p_1)$, there exists a $q'_1 \in A_1$ such that $\{q_1, q'_1, q_2\}$ spoils f_1, f_2.

Case 2: If $k > 2$, there exists an arc from p_1 to p_3 that misses p_2. By the Intermediate Value Theorem, there exists a p_{k+1} on this arc such that $f_k(p_{k+1}) = f_k(p_2)$. However, then $\{p_1, \dots, p_{k+1}\}$ spoils $f_1, \dots, f_k$. This completes the proof.

PROPOSITION 3. *For all* $k, n \geqslant 2$, $CS_k(\mathbb{R}^n, \mathbb{R}) > \lfloor k/2 \rfloor (n-1)$.

Proof. First, note that there is no continuous injection from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$. This follows immediately from the Borsuk–Ulam Theorem [**13**, p. 266]. It suffices to prove that $CS_{2l}(\mathbb{R}^n, \mathbb{R}) > l(n-1)$ for $l \geqslant 1$.

Suppose that we have $l(n-1)$ continuous functions $f_1, \dots, f_{l(n-1)}$ from $\mathbb{R}^n$ to $\mathbb{R}$; we will find $2l$ points that spoil them. For each i, let $F_i: \mathbb{R}^n \to \mathbb{R}^{n-1}$ be defined by

$$F_i(x) = (f_{(i-1)(n-1)+1}(x), \dots, f_{i(n-1)}(x)).$$

As F_i is not injective, there exist x_i, y_i such that $F_i(x_i) = F_i(y_i)$, and thus, for each j, $(i-1)(n-1) < j \leqslant i(n-1), f_j(x_i) = f_j(y_i)$. Then $\{x_i : i = 1, \dots, l\} \cup \{y_i : i = 1, \dots, l\}$ spoils $f_1, \dots, f_{l(n-1)}$. The proof is complete.

We suspect that it should not be too difficult to prove that $CS_k(\mathbb{R}^n, \mathbb{R}) > (k-1)(n-1)$. Showing that the lower bound is $n(k-1)$ may be more difficult.

We conclude with a curious observation. Given n, $CS_k(\mathbb{R}^n, \mathbb{R})$ increases linearly (and hence polynomially, not exponentially) in k. Yet, as we shall see, $CS_{\aleph_0}(\mathbb{R}^n, \mathbb{R})$ is bigger than $\aleph_0$. This suggests that the infinite analogue of this theorem should have $CS_\kappa(\mathbb{R}^n, \mathbb{R})$ increasing slowly above κ, but not so rapidly that $CS_{\aleph_0}(\mathbb{R}^n, \mathbb{R}) = \mathfrak{c}$, where $\mathfrak{c} = 2^{\aleph_0}$. The result is some empirical evidence against the Continuum Hypothesis. (Note that no evidence can be more than empirical, for Cohen [**3**, **4**] proved that the Continuum Hypothesis is neither provable nor disprovable from standard set theory, that is, the Zermelo–Frankel axioms. Whether or not the Continuum Hypothesis is 'true' depends on its utility, philosophical justification, aesthetic appeal, etc. Some authors have argued that, in 'reality', the continuum is 'much' larger than the integers.) Recall that, if $\aleph_\alpha$ is a transfinite cardinal number, then $\aleph_{\alpha+1}$ is the next highest cardinal number.

THEOREM 5. *If* $1 < n < \aleph_0$ *and* $\aleph_\alpha < \mathfrak{c}$, *then* $CS_{\aleph_\alpha}(\mathbb{R}^n, \mathbb{R}) = LS_{\aleph_\alpha}(\mathbb{R}^n, \mathbb{R}) = \aleph_{\alpha+1}$.

Proof. First, $CS_{\aleph_\alpha}(\mathbb{R}^n, \mathbb{R}) > \aleph_\alpha$. Given some continuous functions $f_\beta: \mathbb{R}^n \to \mathbb{R}$, $\beta < \aleph_\alpha$, we choose a sequence of pairs of points p_β, q_β such that, for each β, $f_\beta(p_\beta) = f_\beta(q_\beta)$ and, for any $\beta, \gamma, \{p_\beta, q_\beta, p_\gamma, q_\gamma\}$ contains four distinct points. (This can be done by recursion on β. If p_γ, q_γ have been chosen for all $\gamma < \beta$, then fewer than $\mathfrak{c}$ points have been chosen so far. If f_β is constant, choose any thus far unchosen p_β, q_β; otherwise, choose o_β, r_β such that, for some fixed c, $f_\beta(o_\beta) < c < f_\beta(r_\beta)$, and then imagine $\mathfrak{c}$ mutually disjoint arcs from o_β to r_β. By the Intermediate Value Theorem, each arc contains at least one point p such that $f_\beta(p) = c$, and, as at most $|\beta| < \mathfrak{c}$ of these have been chosen so far, we can choose two more, and call them p_β and q_β.) This set of $\aleph_\alpha$ points spoils $\{f_\beta : \beta < \aleph_\alpha\}$.

On the other hand, $LS_{\aleph_\alpha}(\mathbb{R}^n, \mathbb{R}) \leqslant \aleph_{\alpha+1}$. To prove this, we construct a family of $\aleph_{\alpha+1}$ linear functions from $\mathbb{R}^n$ to $\mathbb{R}$ that separate all $\aleph_\alpha$-subsets of $\mathbb{R}^n$. Let $\mathbb{H}$ be a set of $\aleph_{\alpha+1}$ real numbers. For $r \in \mathbb{H}$, define $f_r: \mathbb{R}^n \to \mathbb{R}$ by setting

$$f_r(x_1, x_2, x_3, \dots, x_n) = x_1 + rx_2 + r^2 x_3 + \dots + r^{n-1} x_n.$$

Let $\mathscr{F} = \{f_r : r \in \mathbb{H}\}$. By Vandermonde's determinant, if $r_1, \ldots, r_n$ are distinct elements of $\mathbb{H}$, then the homogeneous linear system $f_{r_i}(x) = 0$, $1 \leqslant i \leqslant n$, has only the trivial solution $x = 0$. It follows that, if p and q are distinct points of $\mathbb{R}^n$, then the set

$$\mathscr{F}(p,q) = \{f \in \mathscr{F} \mid f(p) = f(q)\}$$

has at most $n-1$ elements. Now we claim that $\mathscr{F}$ separates every set of $\aleph_\alpha$ points. If this is not true, then there is a set S of $\aleph_\alpha$ points that is not separated by $\mathscr{F}$. Hence, for each $f \in \mathscr{F}$, there is a pair of distinct points $p, q \in S$ such that $f(p) = f(q)$. That is,

$$\mathscr{F} = \bigcup_{\{p,q\} \in \binom{S}{2}} \mathscr{F}(p,q) \tag{12}$$

where

$$\binom{S}{2}$$

is the set of all two element subsets of S. However, since

$$\binom{S}{2}$$

has cardinality $\aleph_\alpha$ and each set $\mathscr{F}(p,q)$ has cardinality n, the right-hand side of (12) has cardinality $\aleph_\alpha$, and we have a contradiction.

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